Coordinate Descent Converges Faster with the Gauss-Southwell Rule Than Random Selection

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Abstract

There has been significant recent work on the theory and application of randomized coordinate descent algorithms, beginning with the work of Nesterov [SIAM J. Optim., 22(2), 2012], who showed that a random-coordinate selection rule achieves the same convergence rate as the Gauss-Southwell selection rule. This result suggests that we should never use the Gauss-Southwell rule, because it is typically much more expensive than random selection. However, the empirical behaviours of these algorithms contradict this theoretical result: in applications where the computational costs of the selection rules are comparable, the Gauss-Southwell selection rule tends to perform substantially better than random coordinate selection. We give a simple analysis of the Gauss-Southwell rule showing that—except in extreme cases—its convergence rate is faster than choosing random coordinates. We also (i) show that exact coordinate optimization improves the convergence rate for certain sparse problems, (ii) propose a Gauss-Southwell-Lipschitz rule that gives an even faster convergence rate given knowledge of the Lipschitz constants of the partial derivatives, (iii) analyze the effect of approximate Gauss-Southwell rules, and (iv) analyze proximal-gradient variants of the Gauss-Southwell rule.

1 Coordinate Descent Methods

There has been substantial recent interest in applying coordinate descent methods to solve large-scale optimization problems, starting with the seminal work of Nesterov [2012], who gave the first global rate-of-convergence analysis for coordinate-descent methods for minimizing convex functions. This analysis suggests that choosing a random coordinate to update gives the same performance as choosing the "best" coordinate to update via the more expensive Gauss-Southwell (GS) rule. (Nesterov also proposed a more clever randomized scheme, which we consider later in this paper.) This result gives a compelling argument to use randomized coordinate descent in contexts where the GS rule is too expensive. It also suggests that there is no benefit to using the GS rule in contexts where it is relatively cheap. But in these contexts, the GS rule often substantially outperforms randomized coordinate selection in practice. This suggests that either the analysis of GS is not tight, or that there exists a class of functions for which the GS rule is as slow as randomized coordinate descent.

After discussing contexts in which it makes sense to use coordinate descent and the GS rule, we answer this theoretical question by giving a tighter analysis of the GS rule (under strong-convexity and standard smoothness assumptions) that yields the same rate as the randomized method for a restricted class of functions, but is otherwise faster (and in some cases substantially faster). We further show that, compared to the usual *constant* step-size update of the coordinate, the GS method with exact coordinate optimization has a provably faster rate for problems satisfying a certain sparsity constraint (Section 5). We believe that this is the first result showing a theoretical benefit of exact coordinate optimization; all previous analyses show that these strategies obtain the same rate as constant step-size updates, even though exact optimization tends to be faster in practice. Furthermore, in Section 6, we propose a variant of the GS rule that, similar to Nesterov's more clever randomized sampling scheme, uses knowledge of the Lipschitz constants of the coordinate-wise gradients to obtain a faster rate. We also analyze approximate GS rules (Section 7), which

provide an intermediate strategy between randomized methods and the exact GS rule. Finally, we analyze proximal-gradient variants of the GS rule (Section 8) for optimizing problems that include a separable non-smooth term.

2 Problems of Interest

The rates of Nesterov show that coordinate descent can be faster than gradient descent in cases where, if we are optimizing n variables, the cost of performing n coordinate updates is similar to the cost of performing one full gradient iteration. This essentially means that coordinate descent methods are useful for minimizing convex functions that can be expressed in one of the following two forms:

$$h_1(x) := \sum_{i=1}^n g_i(x_i) + f(Ax), \qquad h_2(x) := \sum_{i \in V} g_i(x_i) + \sum_{(i,j) \in E} f_{ij}(x_i, x_j),$$

where x_i is element i of x, f is smooth and cheap, the f_{ij} are smooth, $G = \{V, E\}$ is a graph, and A is a matrix. (It is assumed that all functions are convex.)¹ The family of functions h_1 includes core machine-learning problems such as least squares, logistic regression, lasso, and SVMs (when solved in dual form) [Hsieh et al., 2008]. Family h_2 includes quadratic functions, graph-based label propagation algorithms for semi-supervised learning [Bengio et al., 2006], and finding the most likely assignments in continuous pairwise graphical models [Rue and Held, 2005].

In general, the GS rule for problem h_2 is as expensive as a full gradient evaluation. However, the structure of G often allows efficient implementation of the GS rule. For example, if each node has at most d neighbours, we can track the gradients of all the variables and use a max-heap structure to implement the GS rule in $O(d \log n)$ time [Meshi et al., 2012]. This is similar to the cost of the randomized algorithm if $d \approx |E|/n$ (since the average cost of the randomized method depends on the average degree). This condition is true in a variety of applications. For example, in spatial statistics we often use two-dimensional grid-structured graphs, where the maximum degree is four and the average degree is slightly less than 4. As another example, for applying graph-based label propagation on the Facebook graph (to detect the spread of diseases, for example), the average number of friends is around 200 but no user has more than seven thousand friends.² The maximum number of friends would be even smaller if we removed edges based on proximity. A non-sparse example where GS is efficient is complete graphs, since here the average degree and maximum degree are both (n-1). Thus, the GS rule is efficient for optimizing dense quadratic functions. On the other hand, GS could be very inefficient for star graphs.

If each column of A has at most c non-zeroes and each row has at most r non-zeroes, then for many notable instances of problem h_1 we can implement the GS rule in $O(cr \log n)$ time by maintaining Ax as well as the gradient and again using a max-heap (see Appendix A). Thus, GS will be efficient if cr is similar to the number of non-zeroes in A divided by n. Otherwise, Dhillon et al. [2011] show that we can approximate the GS rule for problem h_1 with no g_i functions by solving a nearest-neighbour problem. Their analysis of the GS rule in the convex case, however, gives the same convergence rate that is obtained by random selection (although the constant factor can be smaller by a factor of up to n). More recently, Shrivastava and Li [2014] give a general method for approximating the GS rule for problem h_1 with no g_i functions by writing it as a maximum inner-product search problem.

3 Existing Analysis

We are interested in solving the convex optimization problem

$$\min_{x \in \mathbb{P}^n} f(x), \tag{1}$$

 $^{^{1}}$ We could also consider slightly more general cases like functions that are defined on hyper-edges [Richtárik and Takáč, 2015], provided that we can still perform n coordinate updates for a similar cost to one gradient evaluation.

²https://recordsetter.com/world-record/facebook-friends

where ∇f is coordinate-wise L-Lipschitz continuous, i.e., for each $i=1,\ldots,n$,

$$|\nabla_i f(x + \alpha e_i) - \nabla_i f(x)| \le L|\alpha|, \quad \forall x \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{R},$$

where e_i is a vector with a one in position i and zero in all other positions. For twice-differentiable functions, this is equivalent to the assumption that the diagonal elements of the Hessian are bounded in magnitude by L. In contrast, the typical assumption used for gradient methods is that ∇f is L^f -Lipschitz continuous (note that $L \leq L^f \leq Ln$). The coordinate-descent method with constant step-size is based on the iteration

$$x^{k+1} = x^k - \frac{1}{L} \nabla_{i_k} f(x^k) e_{i_k}.$$

The randomized coordinate-selection rule chooses i_k uniformly from the set $\{1, 2, ..., n\}$. Alternatively, the GS rule

$$i_k = \underset{i}{\operatorname{argmax}} |\nabla_i f(x^k)|,$$

chooses the coordinate with the largest directional derivative. Under either rule, because f is coordinate-wise Lipschitz continuous, we obtain the following bound on the progress made by each iteration:

$$f(x^{k+1}) \leq f(x^k) + \nabla_{i_k} f(x^k) (x^{k+1} - x^k)_{i_k} + \frac{L}{2} (x^{k+1} - x^k)_{i_k}^2$$

$$= f(x^k) - \frac{1}{L} (\nabla_{i_k} f(x^k))^2 + \frac{L}{2} \left[\frac{1}{L} \nabla_{i_k} f(x^k) \right]^2$$

$$= f(x^k) - \frac{1}{2L} [\nabla_{i_k} f(x^k)]^2.$$
(2)

We focus on the case where f is μ -strongly convex, meaning that, for some positive μ ,

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} ||y - x||^2, \quad \forall x, y \in \mathbb{R}^n,$$
(3)

which implies that

$$f(x^*) \ge f(x^k) - \frac{1}{2\mu} \|\nabla f(x^k)\|^2,$$
 (4)

where x^* is the optimal solution of (1). This bound is obtained by minimizing both sides of (3) with respect to y.

3.1 Randomized Coordinate Descent

Conditioning on the σ -field \mathcal{F}_{k-1} generated by the sequence $\{x^0, x^1, \dots, x^{k-1}\}$, and taking expectations of both sides of (2), when i_k is chosen with uniform sampling we obtain

$$\begin{split} \mathbb{E}[f(x^{k+1})] &\leq \mathbb{E}\left[f(x^k) - \frac{1}{2L} \left(\nabla_{i_k} f(x^k)\right)^2\right] \\ &= f(x^k) - \frac{1}{2L} \sum_{i=1}^n \frac{1}{n} \left(\nabla_i f(x^k)\right)^2 \\ &= f(x^k) - \frac{1}{2Ln} \|\nabla f(x^k)\|^2. \end{split}$$

Using (4) and subtracting $f(x^*)$ from both sides, we get

$$\mathbb{E}[f(x^{k+1})] - f(x^*) \le \left(1 - \frac{\mu}{Ln}\right) [f(x^k) - f(x^*)]. \tag{5}$$

This is a special of case of Nesterov [2012, Theorem 2] with $\alpha = 0$ in his notation.

3.2 Gauss-Southwell

We now consider the progress implied by the GS rule. By the definition of i_k ,

$$(\nabla_{i_k} f(x^k))^2 = \|\nabla f(x^k)\|_{\infty}^2 \ge (1/n) \|\nabla f(x^k)\|^2.$$
(6)

Applying this inequality to (2), we obtain

$$f(x^{k+1}) \le f(x^k) - \frac{1}{2Ln} \|\nabla f(x^k)\|^2,$$

which together with (4), implies that

$$f(x^{k+1}) - f(x^*) \le \left(1 - \frac{\mu}{Ln}\right) [f(x^k) - f(x^*)].$$
 (7)

This is a special case of Boyd and Vandenberghe [2004, §9.4.3], viewing the GS rule as performing steepest descent in the 1-norm. While this is faster than known rates for cyclic coordinate selection [Beck and Tetruashvili, 2013] and holds deterministically rather than in expectation, this rate is the same as the randomized rate given in (5).

4 Refined Gauss-Southwell Analysis

The deficiency of the existing GS analysis is that too much is lost when we use the inequality in (6). To avoid the need to use this inequality, we instead measure strong-convexity in the 1-norm, i.e.,

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu_1}{2} ||y - x||_1^2,$$

which is the analogue of (3). Minimizing both sides with respect to y, we obtain

$$f(x^*) \ge f(x) - \sup_{y} \{ \langle -\nabla f(x), y - x \rangle - \frac{\mu_1}{2} \|y - x\|_1^2 \}$$

$$= f(x) - \left(\frac{\mu_1}{2} \|\cdot\|_1^2\right)^* (-\nabla f(x))$$

$$= f(x) - \frac{1}{2\mu_1} \|\nabla f(x)\|_{\infty}^2,$$
(8)

which makes use of the convex conjugate $(\frac{\mu_1}{2}\|\cdot\|_1^2)^* = \frac{1}{2\mu_1}\|\cdot\|_{\infty}^2$ [Boyd and Vandenberghe, 2004, §3.3]. Using (8) in (2), and the fact that $(\nabla_{i_k} f(x^k))^2 = \|\nabla f(x^k)\|_{\infty}^2$ for the GS rule, we obtain

$$f(x^{k+1}) - f(x^*) \le \left(1 - \frac{\mu_1}{L}\right) [f(x^k) - f(x^*)]. \tag{9}$$

It is evident that if $\mu_1 = \mu/n$, then the rates implied by (5) and (9) are identical, but (9) is faster if $\mu_1 > \mu/n$. In Appendix B, we show that the relationship between μ and μ_1 can be obtained through the relationship between the squared norms $||\cdot||^2$ and $||\cdot||_1^2$. In particular, we have

$$\frac{\mu}{n} \le \mu_1 \le \mu.$$

Thus, at one extreme the GS rule obtains the same rate as uniform selection ($\mu_1 \approx \mu/n$). However, at the other extreme, it could be faster than uniform selection by a factor of n ($\mu_1 \approx \mu$). This analysis, that the GS rule only obtains the same bound as random selection in an extreme case, supports the better practical behaviour of GS.

4.1 Comparison for Separable Quadratic

We illustrate these two extremes with the simple example of a quadratic function with a diagonal Hessian $\nabla^2 f(x) = \operatorname{diag}((\lambda_1, \dots, \lambda_n))$. In this case,

$$\mu = \min_{i} \lambda_{i}$$
, and $\mu_{1} = \left(\sum_{i=1}^{n} \frac{1}{\lambda_{i}}\right)^{-1}$.

We prove the correctness of this formula for μ_1 in Appendix C. The parameter μ_1 achieves its lower bound when all λ_i are equal, $\lambda_1 = \cdots = \lambda_n = \alpha > 0$, in which case

$$\mu = \alpha$$
 and $\mu_1 = \alpha/n$.

Thus, uniform selection does as well as the GS rule if all elements of the gradient change at *exactly* the same rate. This is reasonable: under this condition, there is no apparent advantage in selecting the coordinate to update in a clever way. Intuitively, one might expect that the favourable case for the Gauss-Southwell rule would be where one λ_i is much larger than the others. However, in this case, μ_1 is again similar to μ/n . To achieve the other extreme, suppose that $\lambda_1 = \beta$ and $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \alpha$ with $\alpha \geq \beta$. In this case, we have $\mu = \beta$ and

$$\mu_1 = \frac{\beta \alpha^{n-1}}{\alpha^{n-1} + (n-1)\beta \alpha^{n-2}} = \frac{\beta \alpha}{\alpha + (n-1)\beta}.$$

If we take $\alpha \to \infty$, then we have $\mu_1 \to \beta$, so $\mu_1 \to \mu$. This case is much less intuitive; GS is n times faster than random coordinate selection if one element of the gradient changes much more *slowly* than the others.

4.2 'Working Together' Interpretation

In the separable quadratic case above, μ_1 is given by the harmonic mean of the eigenvalues of the Hessian divided by n. The harmonic mean is dominated by its smallest values, and this is why having one small value is a notable case. Furthermore, the harmonic mean divided by n has an interpretation in terms of processes 'working together' [Ferger, 1931]. If each λ_i represents the time taken by each process to finish a task (e.g., large values of λ_i correspond to slow workers), then μ is the time needed by the fastest worker to complete the task, and μ_1 is the time needed to complete the task if all processes work together (and have independent effects). Using this interpretation, the GS rule provides the most benefit over random selection when working together is not efficient, meaning that if the n processes work together, then the task is not solved much faster than if the fastest worker performed the task alone. This gives an interpretation of the non-intuitive scenario where GS provides the most benefit: if all workers have the same efficiency, then working together solves the problem n times faster. Similarly, if there is one slow worker (large λ_i), then the problem is solved roughly n times faster by working together. On the other hand, if most workers are slow (many large λ_i), then working together has little benefit.

4.3 Fast Convergence with Bias Term

Consider the standard linear-prediction framework,

$$\underset{x,\beta}{\operatorname{argmin}} \sum_{i=1}^{m} f(a_i^T x + \beta) + \frac{\lambda}{2} ||x||^2 + \frac{\sigma}{2} \beta^2,$$

where we have included a bias variable β (an example of problem h_1). Typically, the regularization parameter σ of the bias variable is set to be much smaller than the regularization parameter λ of the other covariates, to avoid biasing against a global shift in the predictor. Assuming that there is no hidden strong-convexity in the sum, this problem has the structure described in the previous section ($\mu_1 \approx \mu$) where GS has the most benefit over random selection.

5 Rates with Different Lipschitz Constants

Consider the more general scenario where we have a Lipschitz constant L_i for the partial derivative of f with respect to each coordinate i,

$$|\nabla_i f(x + \alpha e_i) - \nabla_i f(x)| \le L_i |\alpha|, \quad \forall x \in \mathbb{R}^n \text{ and } \alpha \in \mathbb{R},$$

and we use a coordinate-dependent step-size at each iteration:

$$x^{k+1} = x^k - \frac{1}{L_{i_k}} \nabla_{i_k} f(x^k) e_{i_k}.$$
(10)

By the logic of (2), in this setting we have

$$f(x^{k+1}) \le f(x^k) - \frac{1}{2L_{i_k}} [\nabla_{i_k} f(x^k)]^2, \tag{11}$$

and thus a convergence rate of

$$f(x^k) - f(x^*) \le \left[\prod_{j=1}^k \left(1 - \frac{\mu_1}{L_{i_j}} \right) \right] [f(x^0) - f(x^*)]. \tag{12}$$

Noting that $L = \max_i \{L_i\}$, we have

$$\prod_{j=1}^{k} \left(1 - \frac{\mu_1}{L_{i_j}} \right) \le \left(1 - \frac{\mu_1}{L} \right)^k. \tag{13}$$

Thus, the convergence rate based on the L_i will be faster, provided that at least one iteration chooses an i_k with $L_{i_k} < L$. In the worst case, however, (13) holds with equality even if the L_i are distinct, as we might need to update a coordinate with $L_i = L$ on every iteration. (For example, consider a separable function where all but one coordinate is initialized at its optimal value, and the remaining coordinate has $L_i = L$.) In Section 6, we discuss selection rules that incorporate the L_i to achieve faster rates whenever the L_i are distinct, but first we consider the effect of exact coordinate optimization on the choice of the L_{i_k} .

5.1 Gauss-Southwell with Exact Optimization

For problems involving functions of the form h_1 and h_2 , we are often able to perform exact (or numerically very precise) coordinate optimization, even if the objective function is not quadratic (e.g., by using a line-search or a closed-form update). Note that (12) still holds when using exact coordinate optimization rather than using a step-size of $1/L_{i_k}$, as in this case we have

$$f(x^{k+1}) = \min_{\alpha} \{ f(x^k + \alpha e_{i_k}) \}$$

$$\leq f\left(x^k - \frac{1}{L_{i_k}} \nabla_{i_i} f(x^k) e_{i_k}\right)$$

$$\leq f(x^k) - \frac{1}{2L_{i_k}} [\nabla_{i_k} f(x^k)]^2,$$
(14)

which is equivalent to (11). However, in practice using exact coordinate optimization leads to better performance. In this section, we show that using the GS rule results in a convergence rate that is indeed faster than (9) for problems with distinct L_i when the function is quadratic, or when the function is not quadratic but we perform exact coordinate optimization.

The key property we use is that, after we have performed exact coordinate optimization, we are guaranteed to have $\nabla_{i_k} f(x^{k+1}) = 0$. Because the GS rule chooses $i_{k+1} = \operatorname{argmax}_i |\nabla_i f(x^{k+1})|$, we cannot have $i_{k+1} = i_k$,

unless x^{k+1} is the optimal solution. Hence, we never choose the same coordinate twice in a row, which guarantees that the inequality (13) is strict (with distinct L_i) and exact coordinate optimization is faster. We note that the improvement may be marginal, as we may simply alternate between the two largest L_i values. However, consider minimizing h_2 when the graph is sparse; after updating i_k , we are guaranteed to have $\nabla_{i_k} f(x^{k+m}) = 0$ for all future iterations (k+m) until we choose a variable i_{k+m-1} that is a neighbour of node i_k in the graph. Thus, if the two largest L_i are not connected in the graph, GS cannot simply alternate between the two largest L_i .

By using this property, in Appendix D we show that the GS rule with exact coordinate optimization for problem h_2 under a chain-structured graph has a convergence rate of the form

$$f(x^k) - f(x^*) \le O\left(\max\{\rho_2^G, \rho_3^G\}^k\right) [f(x^0) - f(x^*)],$$

where ρ_2^G is the maximizer of $\sqrt{(1-\mu_1/L_i)(1-\mu_1/L_j)}$ among all consecutive nodes i and j in the chain, and ρ_3^G is the maximizer of $\sqrt[3]{(1-\mu_1/L_i)(1-\mu_1/L_j)(1-\mu_1/L_k)}$ among consecutive nodes i, j, and k. The implication of this result is that, if the large L_i values are more than two edges from each other in the graph, then we obtain a much better convergence rate. We conjecture that for general graphs, we can obtain a bound that depends on the largest value of ρ_2^G among all nodes i and j connected by a path of length 1 or 2. Note that we can obtain similar results for problem h_1 , by forming a graph that has an edge between nodes i and j whenever the corresponding variables are both jointly non-zero in at least one row of A.

6 Rules Depending on Lipschitz Constants

If the L_i are known, Nesterov [2012] showed that we can obtain a faster convergence rate by sampling proportional to the L_i . We review this result below and compare it to the GS rule, and then propose an improved GS rule for this scenario. Although in this section we will assume that the L_i are known, this assumption can be relaxed using a backtracking procedure [Nesterov, 2012, §6.1].

6.1 Lipschitz Sampling

Taking the expectation of (11) under the distribution $p_i = L_i / \sum_{j=1}^n L_j$ and proceeding as before, we obtain

$$\mathbb{E}[f(x^{k+1})] - f(x^*) \le \left(1 - \frac{\mu}{n\bar{L}}\right) [f(x^k) - f(x^*)],$$

where $\bar{L} = \frac{1}{n} \sum_{j=1}^{n} L_j$ is the average of the Lipschitz constants. This was shown by Leventhal and Lewis [2010] and is a special case of Nesterov [2012, Theorem 2] with $\alpha = 1$ in his notation. This rate is faster than (5) for uniform sampling if any L_i differ.

Under our analysis, this rate may or may not be faster than (9) for the GS rule. On the one extreme, if $\mu_1 = \mu/n$ and any L_i differ, then this Lipschitz sampling scheme is faster than our rate for GS. Indeed, in the context of the problem from Section 4.1, we can make Lipschitz sampling faster than GS by a factor of nearly n by making one λ_i much larger than all the others (recall that our analysis shows no benefit to the GS rule over randomized selection when only one λ_i is much larger than the others). At the other extreme, in our example from Section 4.1 with many large α and one small β , the GS and Lipschitz sampling rates are the same when n = 2, with a rate of $(1 - \beta/(\alpha + \beta))$. However, the GS rate will be faster than the Lipschitz sampling rate for any $\alpha > \beta$ when n > 2, as the Lipschitz sampling rate is $(1 - \beta/((n-1)\alpha + \beta))$, which is slower than the GS rate of $(1 - \beta/(\alpha + (n-1)\beta))$.

6.2 Gauss-Southwell-Lipschitz Rule

Since neither Lipschitz sampling nor GS dominates the other in general, we are motivated to consider if faster rules are possible by combining the two approaches. Indeed, we obtain a faster rate by choosing the

 i_k that minimizes (11), leading to the rule

$$i_k = \underset{i}{\operatorname{argmax}} \frac{|\nabla_i f(x^k)|}{\sqrt{L_i}},$$

which we call the *Gauss-Southwell-Lipschitz* (GSL) rule. Following a similar argument to Section 4, but using (11) in place of (2), the GSL rule obtains a convergence rate of

$$f(x^{k+1}) - f(x^*) \le (1 - \mu_L)[f(x^k) - f(x^*)],$$

where μ_L is the strong-convexity constant with respect to the norm $||x||_L = \sum_{i=1}^n \sqrt{L_i} |x_i|$. This is shown in Appendix E, and in Appendix F we show that

$$\max\left\{\frac{\mu}{n\bar{L}}, \frac{\mu_1}{L}\right\} \le \mu_L \le \frac{\mu_1}{\min_i\{L_i\}}.$$

Thus, the GSL rule is always at least as fast as the fastest of the GS rule and Lipschitz sampling. Indeed, it can be more than a factor of n faster than using Lipschitz sampling, while it can obtain a rate closer to the minimum L_i , instead of the maximum L_i that the classic GS rule depends on.

An interesting property of the GSL rule for quadratic functions is that it is the *optimal* myopic coordinate update. That is, if we have an oracle that can choose the coordinate and the step-size that decreases f by the largest amount, i.e.,

$$f(x^{k+1}) = \underset{i,\alpha}{\operatorname{argmin}} \{ f(x^k + \alpha e_i) \}, \tag{15}$$

this is equivalent to using the GSL rule and the update in (10). This follows because (11) holds with equality in the quadratic case, and the choice $\alpha_k = 1/L_{i_k}$ yields the optimal step-size. Thus, although faster schemes could be possible with non-myopic strategies that cleverly choose the sequence of coordinates or step-sizes, if we can only perform one iteration, then the GSL rule cannot be improved.

For general f, (15) is known as the maximum improvement (MI) rule. This rule has been used in the context of boosting [Rätsch et al., 2001], graphical models [Della Pietra et al., 1997, Lee et al., 2006, Scheinberg and Rish, 2009], Gaussian processes [Bo and Sminchisescu, 2008], and low-rank tensor approximations [Li et al., 2015]. Using an argument similar to (14), our GSL rate also applies to the MI rule, improving existing bounds on this strategy. However, the GSL rule is much cheaper and does not require any special structure (recall that we can estimate L_i as we go).

6.3 Connection between GSL Rule and Normalized Nearest Neighbour Search

Dhillon et al. [2011] discuss an interesting connection between the GS rule and the nearest-neighbour-search (NNS) problem for objectives of the form

$$\min_{x \in \mathbb{R}^n} F(x) = f(Ax),\tag{16}$$

This is a special case of h_1 with no g_i functions, and its gradient has the special form

$$\nabla F(x) = A^T r(x),$$

where $r(x) = \nabla f(Ax)$. We use the symbol r because it is the residual vector (Ax - b) in the special case of least squares. For this problem structure the GS rule has the form

$$i_k = \underset{i}{\operatorname{argmax}} |\nabla_i f(x^k)|$$
$$= \underset{i}{\operatorname{argmax}} |r(x^k)^T a_i|,$$

where a_i denotes column i of A for i = 1, ..., n. Dhillon et al. [2011] propose to approximate the above argmax by solving the following NNS problem

$$i_k = \operatorname*{argmin}_{i \in [2n]} ||r(x^k) - a_i||,$$

where i in the range (n+1) through 2n refers to the negation $-(a_{i-n})$ of column (i-n) and if the selected i_k is greater than n we return (i-n). We can justify this approximation using the logic

$$\begin{split} i_k &= \operatorname*{argmin}_{i \in [2n]} \| r(x^k) - a_i \| \\ &= \operatorname*{argmin}_{i \in [2n]} \frac{1}{2} \| r(x^k) - a_i \|^2 \\ &= \operatorname*{argmin}_{i \in [2n]} \underbrace{\frac{1}{2} \| r(x^k) \|^2}_{\text{constant}} - r(x^k)^T a_i + \frac{1}{2} \| a_i \|^2 \\ &= \operatorname*{argmax}_{i \in [2n]} r(x^k)^T a_i - \frac{1}{2} \| a_i \|^2 \\ &= \operatorname*{argmax}_{i \in [n]} | r(x^k)^T a_i | - \frac{1}{2} \| a_i \|^2 \\ &= \operatorname*{argmax}_{i \in [n]} | \nabla_i f(x^k) | - \frac{1}{2} \| a_i \|^2. \end{split}$$

Thus, the NNS computes an approximation to the GS rule that is biased towards coordinates where $||a_i||$ is small. Note that this formulation is equivalent to the GS rule in the special case that $||a_i|| = 1$ (or any other constant) for all i. Shrivastava and Li [2014] have more recently considered the case where $||a_i|| \le 1$ and incorporate powers of $||a_i||$ in the NNS to yield a better approximation.

A further interesting property of the GSL rule is that we can often formulate the exact GSL rule as a normalized NNS problem. In particular, for problem (16) the Lipschitz constants will often have the form $L_i = \gamma ||a_i||^2$ for a some positive scalar γ . For example, least squares has $\gamma = 1$ and logistic regression has $\gamma = 0.25$. When the Lipschitz constants have this form, we can compute the exact GSL rule by solving a normalized NNS problem,

$$i_k = \underset{i \in [2n]}{\operatorname{argmin}} \left\| r(x^k) - \frac{a_i}{\|a_i\|} \right\|. \tag{17}$$

The exactness of this formula follows because

$$i_{k} = \underset{i \in [2n]}{\operatorname{argmin}} \left\| r(x^{k}) - \frac{a_{i}}{\|a_{i}\|} \right\|$$

$$= \underset{i \in [2n]}{\operatorname{argmin}} \frac{1}{2} \| r(x^{k}) - a_{i} / \|a_{i}\| \|^{2}$$

$$= \underset{i \in [2n]}{\operatorname{argmin}} \frac{1}{2} \| r(x^{k}) \|^{2} - \frac{r(x^{k})^{T} a_{i}}{\|a_{i}\|} + \underbrace{\frac{1}{2} \frac{\|a_{i}\|^{2}}{\|a_{i}\|^{2}}}_{\text{constant}}$$

$$= \underset{i \in [n]}{\operatorname{argmax}} \frac{|r(x^{k})^{T} a_{i}|}{\|a_{i}\|}$$

$$= \underset{i \in [n]}{\operatorname{argmax}} \frac{|r(x^{k})^{T} a_{i}|}{\sqrt{\gamma} \|a_{i}\|}$$

$$= \underset{i \in [n]}{\operatorname{argmax}} \frac{|\nabla_{i} f(x^{k})|}{\sqrt{L_{i}}}.$$

Thus, the form of the Lipschitz constant conveniently removes the bias towards smaller values of $||a_i||$ that gets introduced when we try to formulate the classic GS rule as a NNS problem. Interestingly, in this setting we do not need to know γ to implement the GSL rule as a NNS problem.

7 Approximate Gauss-Southwell

In many applications, computing the exact GS rule is too inefficient to be of any practical use. However, a computationally cheaper approximate GS rule might be available. Approximate GS rules under multiplicative and additive errors were considered by Dhillon et al. [2011] in the convex case, but in this setting the convergence rate is similar to the rate achieved by random selection. In this section, we give rates depending on μ_1 for approximate GS rules.

7.1 Multiplicative Errors

In the multiplicative error regime, the approximate GS rule chooses an i_k satisfying

$$|\nabla_{i_k} f(x^k)| \ge ||\nabla f(x^k)||_{\infty} (1 - \epsilon_k),$$

for some $\epsilon_k \in [0, 1)$. In this regime, our basic bound on the progress (2) still holds, as it was defined for any i_k . We can incorporate this type of error into our lower bound (8) to obtain

$$f(x^*) \ge f(x^k) - \frac{1}{2\mu_1} \|\nabla f(x^k)\|_{\infty}^2$$

$$\ge f(x^k) - \frac{1}{2\mu_1 (1 - \epsilon_k)^2} |\nabla_{i_k} f(x^k)|^2.$$

This implies a convergence rate of

$$f(x^{k+1}) - f(x^*) \le \left(1 - \frac{\mu_1(1 - \epsilon_k)^2}{L}\right) [f(x^k) - f(x^*)].$$

Thus, the convergence rate of the method is nearly identical to using the exact GS rule for small ϵ_k (and it degrades gracefully with ϵ_k). This is in contrast to having an error in the gradient [Friedlander and Schmidt, 2012], where the error ϵ must decrease to zero over time.

7.2 Additive Errors

In the additive error regime, the approximate GS rule chooses an i_k satisfying

$$|\nabla_{i_k} f(x^k)| \ge ||\nabla f(x^k)||_{\infty} - \epsilon_k,$$

for some $\epsilon_k \geq 0$. In Appendix G, we show that under this rule, we have

$$f(x^k) - f(x^*) \le \left(1 - \frac{\mu_1}{L}\right)^k \left[f(x^0) - f(x^*) + A_k\right],$$

where

$$A_k \le \min \left\{ \sum_{i=1}^k \left(1 - \frac{\mu_1}{L} \right)^{-i} \epsilon_i \frac{\sqrt{2L_1}}{L} \sqrt{f(x^0) - f(x^*)}, \ \sum_{i=1}^k \left(1 - \frac{\mu_1}{L} \right)^{-i} \left(\epsilon_i \sqrt{\frac{2}{L}} \sqrt{f(x^0) - f(x^*)} + \frac{\epsilon_i^2}{2L} \right) \right\},$$

where L_1 is the Lipschitz constant of ∇f with respect to the 1-norm. Note that L_1 could be substantially larger than L, so the second part of the maximum in A_k is likely to be the smaller part unless the ϵ_i are large. This regime is closer to the case of having an error in the gradient, as to obtain convergence the ϵ_k

must decrease to zero. This result implies that a sufficient condition for the algorithm to obtain a linear convergence rate is that the errors ϵ_k converge to zero at a linear rate. Further, if the errors satisfy $\epsilon_k = O(\rho^k)$ for some $\rho < (1 - \mu_1/L)$, then the convergence rate of the method is the same as if we used an exact GS rule. On the other hand, if ϵ_k does not decrease to zero, we may end up repeatedly updating the same wrong coordinate and the algorithm will not converge (though we could switch to the randomized method if this is detected).

8 Proximal-Gradient Gauss-Southwell

One of the key motivations for the resurgence of interest in coordinate descent methods is their performance on problems of the form

$$\min_{x \in \mathbb{R}^n} F(x) \equiv f(x) + \sum_{i=1}^n g_i(x_i),$$

where f is smooth and convex and the g_i are convex, but possibly non-smooth. This includes problems with ℓ_1 -regularization, and optimization with lower and/or upper bounds on the variables. Similar to proximal-gradient methods, we can apply the proximal operator to the coordinate update,

$$x^{k+1} = \operatorname{prox}_{\frac{1}{L}g_{i_k}} \left[x^k - \frac{1}{L} \nabla_{i_k} f(x^k) e_{i_k} \right],$$

where

$$\operatorname{prox}_{\alpha g_i}[y] = \operatorname*{argmin}_{x \in \mathbb{R}^n} \frac{1}{2} ||x - y||^2 + \alpha g_i(x).$$

With random coordinate selection, Richtárik and Takáč [2014] show that this method has a convergence rate of

$$\mathbb{E}[F(x^{k+1}) - F(x^*)] \le \left(1 - \frac{\mu}{nL}\right) [F(x^k) - F(x^*)],$$

similar to the unconstrained/smooth case.

There are several generalizations of the GS rule to this scenario. Here we consider three possibilities, all of which are equivalent to the GS rule if the g_i are not present. First, the GS-s rule chooses the coordinate with the most negative directional derivative. This strategy is popular for ℓ_1 -regularization [Shevade and Keerthi, 2003, Wu and Lange, 2008, Li and Osher, 2009] and in general is given by [see Bertsekas, 1999, §8.4]

$$i_k = \underset{i}{\operatorname{argmax}} \left\{ \min_{s \in \partial g_i} |\nabla_i f(x^k) + s| \right\}.$$

However, the length of the step $(\|x^{k+1} - x^k\|)$ could be arbitrarily small under this choice. In contrast, the GS-r rule chooses the coordinate that maximizes the length of the step [Tseng and Yun, 2009, Dhillon et al., 2011],

$$i_k = \underset{i}{\operatorname{argmax}} \left\{ \left| x_i^k - \operatorname{prox}_{\frac{1}{L}g_i} \left[x_i^k - \frac{1}{L} \nabla_i f(x^k) \right] \right| \right\}.$$

This rule is effective for bound-constrained problems, but it ignores the change in the non-smooth term $(g_i(x_i^{k+1}) - g_i(x_k^k))$. Finally, the GS-q rule maximizes progress assuming a quadratic upper bound on f [Tseng and Yun, 2009],

$$i_k = \underset{i}{\operatorname{argmin}} \left\{ \min_{d} \left\{ f(x^k) + \nabla_i f(x^k) d + \frac{L}{2} d^2 + g_i(x_i^k + d) - g_i(x_i^k) \right\} \right\}.$$

While the least intuitive rule, the GS-q rule seems to have the best theoretical properties. Further, if we use L_i in place of L in the GS-q rule (which we call the GSL-q strategy), then we obtain the GSL rule if the g_i

are not present. In contrast, using L_i in place of L in the GS-r rule (which we call the GSL-r strategy) does not yield the GSL rule as a special case.

In Appendix H, we show that using the GS-q rule yields a convergence rate of

$$F(x^{k+1}) - F(x^*) \le \min \left\{ \left(1 - \frac{\mu}{Ln} \right) [f(x^k) - f(x^*)], \left(1 - \frac{\mu_1}{L} \right) [f(x^k) - f(x^*)] + \epsilon_k \right\},$$

where ϵ_k is bounded above by a measure of the non-linearity of the g_i along the possible coordinate updates times the inverse condition number μ_1/L . Note that ϵ_k goes to zero as k increases and we conjecture that the above bound actually holds with $\epsilon_k = 0$. In contrast, in Appendix H we also give counter-examples showing that the above rate does not hold with $\epsilon_k = 0$ for the GS-s or GS-r rule, even if you replace the minimum by a maximum. Thus, any bound for the GS-s or GS-r rule would be slower than the expected rate under random selection, while the GS-q rule leads to a better bound.

9 Experiments

We first compare the efficacy of different coordinate selection rules on the following simple instances of h_1 . ℓ_2 -regularized sparse least squares: Here we consider the problem

$$\min_{x} \frac{1}{2n} ||Ax - b||^2 + \frac{\lambda}{2} ||x||^2,$$

an instance of problem h_1 . We set A to be an m by n matrix with entries sampled from a $\mathcal{N}(0,1)$ distribution (with m=1000 and n=1000). We then added 1 to each entry (to induce a dependency between columns), multiplied each column by a sample from $\mathcal{N}(0,1)$ multiplied by ten (to induce different Lipschitz constants across the coordinates), and only kept each entry of A non-zero with probability $10 \log(n)/n$ (a sparsity level that allows the Gauss-Southwell rule to be applied with cost $O(\log^3(n))$). We set $\lambda=1$ and b=Ax+e, where the entries of x and e were drawn from a $\mathcal{N}(0,1)$ distribution. In this setting, we used a step-size of $1/L_i$ for each coordinate i, which corresponds to exact coordinate optimization.

 ℓ_2 -regularized sparse logistic regression: Here we consider the problem

$$\min_{x} \frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-b_i a_i^T x)) + \frac{\lambda}{2} ||x||^2.$$

We set the a_i^T to be the rows of A from the previous problem, and set b = sign(Ax), but randomly flipping each b_i with probability 0.1. In this setting, we compared using a step-size of $1/L_i$ to using exact coordinate optimization.

Over-determined dense least squares: Here we consider the problem

$$\min_{x} \frac{1}{2n} ||Ax - b||^2,$$

but, unlike the previous case, we do not set elements of A to zero and we make A have dimension 1000 by 100. Because the system is over-determined, it does not need an explicit strongly-convex regularizer to induce global strong-convexity. In this case, the density level means that the exact GS rule is not efficient. Hence, we use a balltree structure [Omohundro, 1989] to implement an efficient approximate GS rule based on the connection to the NNS problem discovered by Dhillon et al. [2011]. On the other hand, we can compute the exact GSL rule for this problem as a NNS problem as discussed in Section 6.3.

 ℓ_1 -regularized underdetermined sparse least squares: Here we consider the non-smooth problem

$$\min_{x} \frac{1}{2n} ||Ax - b||^2 + \lambda ||x||_1.$$

We generate A as we did for the ℓ_2 -regularized sparse least squares problem, except with the dimension 1000 by 10000. This problem is not globally strongly-convex, but will be strongly-convex along the dimensions that are non-zero in the optimal solution.

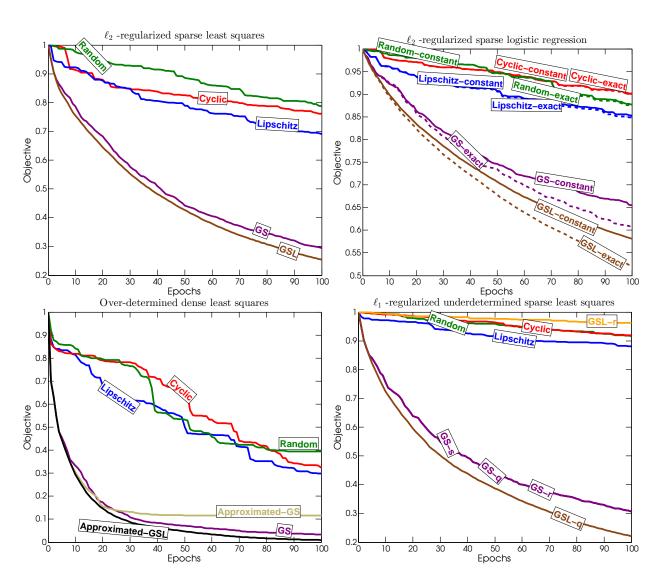


Figure 1: Comparison of coordinate selection rules for 4 instances of problem h_1 .

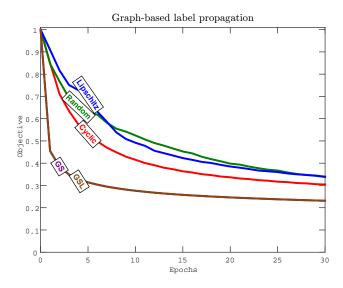


Figure 2: Comparison of coordinate selection rules for graph-based semi-supervised learning.

We plot the objective function (divided by its initial value) of coordinate descent under different selection rules in Figure 1. Even on these simple datasets, we see dramatic differences in performance between the different strategies. In particular, the GS rule outperforms random coordinate selection (as well as cyclic selection) by a substantial margin in all cases. The Lipschitz sampling strategy can narrow this gap, but it remains large (even when an approximate GS rule is used). The difference between GS and randomized selection seems to be most dramatic for the ℓ_1 -regularized problem; the GS rules tend to focus on the non-zero variables while most randomized/cyclic updates focus on the zero variables, which tend not to move away from zero.³ Exact coordinate optimization and using the GSL rule seem to give modest but consistent improvements. The three non-smooth GS-* rules had nearly identical performance despite their different theoretical properties. The GSL-q rule gave better performance than the GS-* rules, while the the GSL-r variant performed worse than even cyclic and random strategies. We found it was also possible to make the GS-r rule perform poorly by perturbing the initialization away from zero. While these experiments plot the performance in terms of the number of iterations, in Appendix I we show that the GS-r rules can also be advantageous in terms of runtime.

We next consider an instance of problem h_2 , performing label propagation for semi-supervised learning in the 'two moons' dataset [Zhou et al., 2004]. We generate 500 samples from this dataset, randomly label five points in the data, and connect each node to its five nearest neighbours. This high level of sparsity is typical of graph-based methods for semi-supervised learning, and allows the exact Gauss-Southwell rule to be implemented efficiently. We use the quadratic labeling criterion of Bengio et al. [2006], which allows exact coordinate optimization and is normally optimized with cyclic coordinate descent. We plot the performance under different selection rules in Figure 2. Here, we see that even cyclic coordinate descent outperforms randomized coordinate descent, but that the GS and GSL rules give even better performance. We note that the GS and GSL rules perform similarly on this problem since the Lipschitz constants do not vary much.

10 Discussion

It is clear that the GS rule is not practical for every problem where randomized methods are applicable. Nevertheless, we have shown that even approximate GS rules can obtain better convergence rate bounds than fully-randomized methods. We have given a similar justification for the use of exact coordinate optimization,

 $^{^3}$ To reduce the cost of the GS-s method in this context, Shevade and Keerthi [2003] consider a variant where we first compute the GS-s rule for the non-zero variables and if an element is sufficiently large then they do not consider the zero variables.

and we note that our argument could also be used to justify the use of exact coordinate optimization within randomized coordinate descent methods (as used in our experiments). We have also proposed the improved GSL rule, and considered approximate/proximal variants. We expect our analysis also applies to block updates by using mixed norms $\|\cdot\|_{p,q}$, and could be used for accelerated/parallel methods [Fercoq and Richtárik, 2013], for primal-dual rates of dual coordinate ascent [Shalev-Shwartz and Zhang, 2013], for successive projection methods [Leventhal and Lewis, 2010], for boosting algorithms [Rätsch et al., 2001], and for scenarios without strong-convexity under general error bounds [Luo and Tseng, 1993].

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Appendix A Efficient calculation of GS rules for sparse problems

We first give additional details on how to calculate the GS rule efficiently for sparse instances of problems h_1 and h_2 . We will consider the case where each g_i is smooth, but the ideas can be extended to allow a non-smooth g_i . Further, note that the efficient calculation does not rely on convexity, so these strategies can also be used for non-convex problems.

A.1 Problem h_2

Problem h_2 has the form

$$h_2(x) := \sum_{i \in V} g_i(x_i) + \sum_{(i,j) \in E} f_{ij}(x_i, x_j),$$

where each g_i and f_{ij} are differentiable and $G = \{V, E\}$ is a graph where the number of vertices |V| is the same as the number of variables n. If all nodes in the graph have a degree (number of neighbours) bounded above by some constant d, we can implement the GS rule in $O(d \log n)$ after an O(n + |E|) time initialization by maintaining the following information about x^k :

- 1. A vector containing the values $\nabla_i g_i(x_i^k)$.
- 2. A matrix containing the values $\nabla_i f_{ij}(x_i^k, x_i^k)$ in the first column and $\nabla_j f_{ij}(x_i^k, x_i^k)$ in the second column.
- 3. The elements of the gradient vector $\nabla h_2(x^k)$ stored in a binary max heap data structure [see Cormen et al., 2001, Chapter 6].

Given the heap structure, we can compute the GS rule in O(1) by simply reading the index value of the root node in the max heap. The costs for initializing these structures are:

- 1. O(n) to compute $g_i(x_i^0)$ for all n nodes.
- 2. O(|E|) to compute $\nabla_{ij} f_{ij}(x_i^0, x_i^0)$ for all |E| edges.
- 3. O(n + |E|) to sum the values in the above structures to compute $\nabla h(x^0)$, and O(n) to construct the initial max heap.

Thus, the one-time initialization cost is O(n + |E|). The costs of updating the data structures after we update $x_{i_k}^k$ to $x_{i_k}^{k+1}$ for the selected coordinate i_k are:

- 1. O(1) to compute $g_{i_k}(x_{i_k}^{k+1})$.
- 2. O(d) to compute $\nabla_{ij} f_{ij}(x_i^{k+1}, x_j^{k+1})$ for $(i, j) \in E$ and $i = i_k$ or $j = i_k$ (only d such values exist by assumption, and all other $\nabla_{ij} f_{ij}(x_i, x_j)$ are unchanged).

3. O(d) to update up to d elements of $\nabla h(x^{k+1})$ that differ from $\nabla h(x^k)$ by using differences in changed values of g_i and f_{ij} , followed by $O(d \log n)$ to perform d updates of the heap at a cost of $O(\log n)$ for each update.

The most expensive part of the update is modifying the heap, and thus the total cost is $O(d \log n)$.

A.2 Problem h_1

Problem h_1 has the form

$$h_1(x) := \sum_{i=1}^n g_i(x_i) + f(Ax),$$

where g_i and f are differentiable, and A is an m by n matrix where we denote column i by a_i and row j by a_j^T . Note that f is a function from \mathbb{R}^m to \mathbb{R} , and we assume $\nabla_j f$ only depends on $a_j^T x$. While this is a strong assumption (e.g., it rules out f being the product function), this class includes a variety of notable problems like the least squares and logistic regression models from our experiments. If A has z non-zero elements, with a maximum of c non-zero elements in each column and r non-zero elements in each row, then with a pre-processing cost of O(z) we can implement the GS rule in this setting in $O(cr \log n)$ by maintaining the following information about x^k :

- 1. A vector containing the values $\nabla_i g_i(x_i^k)$.
- 2. A vector containing the product Ax^k .
- 3. A vector containing the values $\nabla f(Ax^k)$.
- 4. A vector containing the product $A^T \nabla f(Ax^k)$
- 5. The elements of the gradient vector $\nabla h_1(x^k)$ stored in a binary max heap data structure.

The heap structure again allows us to compute the GS rule in O(1), and the costs of initializing these structures are:

- 1. O(n) to compute $q_i(x_i^0)$ for all n variables.
- 2. O(z) to compute the product Ax^0 .
- 3. O(m) to compute $\nabla f(Ax^0)$ (using that $\nabla_j f$ only depends on $a_i^T x^0$).
- 4. O(z) to compute $A^T \nabla f(Ax^0)$.
- 5. O(n) to add the $\nabla_i g_i(x_i^0)$ to the above product to obtain $\nabla h_1(x^0)$ and construct the initial max heap.

As it is reasonable to assume that $z \ge m$ and $z \ge n$ (e.g., we have at least one non-zero in each row and column), the cost of the initialization is thus O(z). The costs of updating the data structures after we update $x_{i_k}^k$ to $x_{i_k}^{k+1}$ for the selected coordinate i_k are:

- 1. O(1) to compute $g_{i_k}(x_{i_k}^{k+1})$.
- 2. O(c) to update the product using $Ax^{k+1} = Ax^k + (x_{i_k}^{k+1} x_{i_k}^k)a_i$, since a_i has at most c non-zero values.
- 3. O(c) to update up to c elements of $\nabla f(Ax^{k+1})$ that have changed (again using that $\nabla_j f$ only depends on $a_i^T x^{k+1}$).

⁴For less-sparse problems where $n < d \log n$, using a heap is actually inefficient and we should simply store $\nabla h(x^k)$ as a vector. The initialization cost is the same, but we can then perform the GS rule in O(n) by simply searching through the vector for the maximum element.

- 4. O(cr) to perform up to c updates of the form $A^T \nabla f(Ax^{k+1}) = A^T \nabla f(Ax^k) + (\nabla_j f(Ax^{k+1}) \nabla_j f(Ax^k))(a_i)^T$, where each update costs O(r) since each a_i has at most r non-zero values.
- 5. $O(cr \log n)$ to update the gradients in the heap.

The most expensive part is again the heap update, and thus the total cost is $O(cr \log n)$.

Appendix B Relationship between μ_1 and μ

We can establish the relationship between μ and μ_1 by using the known relationship between the 2-norm and the 1-norm,

$$||x||_1 \ge ||x|| \ge \frac{1}{\sqrt{n}} ||x||_1.$$

In particular, if we assume that f is μ -strongly convex in the 2-norm, then for all x and y we have

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} ||y - x||^2$$

$$\ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2n} ||y - x||_1^2,$$

implying that f is at least $\frac{\mu}{n}$ -strongly convex in the 1-norm. Similarly, if we assume that a given f is μ_1 -strongly convex in the 1-norm then for all x and y we have

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu_1}{2} ||y - x||_1^2$$

$$\ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu_1}{2} ||y - x||^2,$$

implying that f is at least μ_1 -strongly convex in the 2-norm. Summarizing these two relationships, we have

$$\frac{\mu}{n} \le \mu_1 \le \mu$$
.

Appendix C Analysis for separable quadratic case

We first establish an equivalent definition of strong-convexity in the 1-norm, along the lines of Nesterov [2004, Theorem 2.1.9]. Subsequently, we use this equivalent definition to derive μ_1 for a separable quadratic function.

C.1 Equivalent definition of strong-convexity

Assume that f is μ_1 -strongly convex in the 1-norm, so that for any $x, y \in \mathbb{R}^n$ we have

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu_1}{2} ||y - x||_1^2.$$

Reversing x and y in the above gives

$$f(x) \ge f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu_1}{2} ||x - y||_1^2,$$

and adding these two together yields

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \ge \mu_1 \|y - x\|_1^2. \tag{18}$$

Conversely, assume that for all x and y we have

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \ge \mu_1 \|y - x\|_1^2$$

and consider the function $g(\tau) = f(x + \tau(y - x))$ for $\tau \in \mathbb{R}$. Then

$$\begin{split} f(y) - f(x) - \langle \nabla f(x), y - x \rangle &= g(1) - g(0) - \langle \nabla f(x), y - x \rangle \\ &= \int_0^1 \frac{dg}{d\tau}(\tau) - \langle \nabla f(x), y - x \rangle \ d\tau \\ &= \int_0^1 \langle \nabla f(x + \tau(y - x)), y - x \rangle - \langle \nabla f(x), y - x \rangle \ d\tau \\ &= \int_0^1 \langle \nabla f(x + \tau(y - x)) - \nabla f(x), y - x \rangle \ d\tau \\ &\geq \int_0^1 \frac{\mu_1}{\tau} \|\tau(y - x)\|_1^2 \ d\tau \\ &= \int_0^1 \mu_1 \tau \|y - x\|_1^2 \ d\tau \\ &= \frac{\mu_1}{2} \tau^2 \|y - x\|_1^2 \bigg|_0^1 \\ &= \frac{\mu_1}{2} \|y - x\|_1^2. \end{split}$$

Thus, μ_1 -strong convexity in the 1-norm is equivalent to having

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \ge \mu_1 \|y - x\|_1^2 \quad \forall \ x, y. \tag{19}$$

C.2 Strong-convexity constant μ_1 for separable quadratic functions

Consider a strongly convex quadratic function f with a diagonal Hessian $H = \nabla^2 f(x) = \text{diag}(\lambda_1, \dots, \lambda_n)$, where $\lambda_i > 0$ for all $i = 1, \dots, n$. We show that in this case

$$\mu_1 = \left(\sum_{i=1}^n \frac{1}{\lambda_i}\right)^{-1}.$$

From the previous section, μ_1 is the minimum value such that (19) holds,

$$\mu_1 = \inf_{x \neq y} \frac{\langle \nabla f(y) - \nabla f(x), y - x \rangle}{\|y - x\|_1^2}.$$

Using $\nabla f(x) = Hx + b$ for some b and letting z = y - x, we get

$$\begin{split} \mu_1 &= \inf_{x \neq y} \frac{\langle (Hy - b) - (Hx - b), y - x \rangle}{\|y - x\|_1^2} \\ &= \inf_{x \neq y} \frac{\langle H(y - x), y - x \rangle}{\|y - x\|_1^2} \\ &= \inf_{z \neq 0} \frac{z^T H z}{\|z\|_1^2} \\ &= \min_{\|z\|_1 = 1} z^T H z \\ &= \min_{e^T z = 1} \sum_{i=1}^n \lambda_i z_i^2, \end{split}$$

where the last two lines use that the objective is invariant to scaling of z and to the sign of z (respectively), and where e is a vector containing a one in every position. This is an equality-constrained strictly-convex

quadratic program, so its solution is given as a stationary point (z^*, η^*) of the Lagrangian,

$$\Lambda(z,\eta) = \sum_{i=1}^{n} \lambda_i z_i^2 + \eta(1 - e^T z).$$

Differentiating with respect to each z_i for $i=1,\ldots,n$ and equating to zero, we have for all i that $2\lambda_i z_i^* - \eta^* = 0$, or

$$z_i^* = \frac{\eta^*}{2\lambda_i}. (20)$$

Differentiating the Lagrangian with respect to η and equating to zero we obtain $1 - e^T z^* = 0$, or equivalently

$$1 = e^T z^* = \frac{\eta^*}{2} \sum_j \frac{1}{\lambda_j},$$

which yields

$$\eta^* = 2 \left(\sum_j \frac{1}{\lambda_j} \right)^{-1}.$$

Combining this result for η^* with equation (20), we have

$$z_i^* = \frac{1}{\lambda_i} \left(\sum_j \frac{1}{\lambda_j} \right)^{-1}.$$

This gives the minimizer, so we evaluate the objective at this point to obtain μ_1 ,

$$\mu_1 = \sum_{i=1}^n \lambda_i (z_i^*)^2$$

$$= \sum_{i=1}^n \lambda_i \left(\frac{1}{\lambda_i} \left(\sum_{j=1}^n \frac{1}{\lambda_j} \right)^{-1} \right)^2$$

$$= \sum_{i=1}^n \frac{1}{\lambda_i} \left(\sum_{j=1}^n \frac{1}{\lambda_j} \right)^{-2}$$

$$= \left(\sum_{j=1}^n \frac{1}{\lambda_j} \right)^{-2} \left(\sum_{i=1}^n \frac{1}{\lambda_i} \right)$$

$$= \left(\sum_{j=1}^n \frac{1}{\lambda_j} \right)^{-1}.$$

Appendix D Gauss-Southwell with exact optimization

We can obtain a faster convergence for GS using exact coordinate optimization for sparse variants of problems h_1 and h_2 , by observing that the convergence rate can be expressed in terms of the sequence of $(1 - \mu_1/L_{i_k})$ values,

$$f(x^k) - f(x^*) \le \left[\prod_{j=1}^k \left(1 - \frac{\mu_1}{L_{i_j}} \right) \right] [f(x^0) - f(x^*)].$$

The worst case occurs when the product of the $(1 - \mu_1/L_{i_k})$ values is as large as possible. However, using exact coordinate optimization guarantees that, after we have updated coordinate i, the GS rule will never select it again until one of its neighbours has been selected. Thus, we can obtain a tighter bound on the worst-case convergence rate using GS with exact coordinate optimization on iteration k, by solving the following combinatorial optimization problem defined on a weighted graph:

Problem 1. We are given a graph G = (V, E) with n nodes, a number M_i associated with each node i, and an iteration number k. Choose a sequence $\{i_t\}_{t=1}^k$ that maximizes the sum of the M_{i_t} , subject to the following constraint: after each time node i has been chosen, it cannot be chosen again until after a neighbour of node i has been chosen.

We can use the M_i chosen by this problem to obtain an upper-bound on the sequence of $\log(1-\mu_1/L_i)$ values, and if the largest M_i values are not close to each other in the graph, then this rate can be much faster than the rate obtained by alternating between the largest M_i values. In the particular case of chain-structured graphs, a worst-case sequence can be constructed that spends all but O(n) iterations in one of two solution modes: (i) alternate between two nodes i and j that are connected by an edge with the highest value of $\frac{M_i+M_j}{2}$, or (ii) alternate between three nodes $\{i,j,k\}$ with the highest value of $\frac{M_i+M_j+M_k}{3}$, where there is an edge from i to j and from j to k, but not from i to k. To show that these are the two solution modes, observe that the solution must eventually cycle because there are a finite number of nodes. If you have more than three nodes in the cycle, then you can always remove one node from the cycle to obtain a better average weight for the cycle without violating the constraint. We will fall into mode (i) if the average of M_i and M_j in this mode is larger than the average of M_i , M_j and M_k in the second mode. We can construct a solution to this problem that consists of a 'burn-in' period, where we choose the largest M_i , followed by repeatedly going through the better of the two solution modes up until the final three steps, where a 'burn-out' phase arranges to finish with several large M_i . By setting $M_i = \log(1 - \mu_1/L_i)$, this leads to a convergence rate of the form

$$f(x^k) - f(x^*) \le O\left(\max\{\rho_2^G, \rho_3^G\}^k\right) [f(x^0) - f(x^*)],$$

where ρ_2^G is the maximizer of $\sqrt{(1-\mu_1/L_i)(1-\mu_1/L_j)}$ among all consecutive nodes i and j in the chain, and ρ_3^G is the maximizer of $\sqrt[3]{(1-\mu_1/L_i)(1-\mu_1/L_j)(1-\mu_1/L_k)}$ among consecutive nodes i, j, and k. The O() notation gives the constant due to choosing higher $(1-\mu_1/L_i)$ values during the burn-in and burn-out periods. The implication of this result is that, if the large L_i values are more than two edges away from each other in the graph, then the convergence rate can be much faster.

Appendix E Gauss-Southwell-Lipschitz rule: convergence rate

The coordinate-descent method with a constant step-size of L_{i_k} uses the iteration

$$x^{k+1} = x^k - \frac{1}{L_{i_k}} \nabla_{i_k} f(x^k) e_{i_k}.$$

Because f is coordinate-wise L_{i_k} -Lipschitz continuous, we obtain the following bound on the progress made by each iteration:

$$f(x^{k+1}) \leq f(x^k) + \nabla_{i_k} f(x^k) (x^{k+1} - x^k)_{i_k} + \frac{L_{i_k}}{2} (x^{k+1} - x^k)_{i_k}^2$$

$$= f(x^k) - \frac{1}{L_{i_k}} (\nabla_{i_k} f(x^k))^2 + \frac{L_{i_k}}{2} \left[\frac{1}{L_{i_k}} \nabla_{i_k} f(x^k) \right]^2$$

$$= f(x^k) - \frac{1}{2L_{i_k}} [\nabla_{i_k} f(x^k)]^2$$

$$= f(x^k) - \frac{1}{2} \left[\frac{\nabla_{i_k} f(x^k)}{\sqrt{L_{i_k}}} \right]^2.$$
(21)

By choosing the coordinate to update according to the Gauss-Southwell-Lipchitz (GSL) rule,

$$i_k = \underset{i}{\operatorname{argmax}} \frac{|\nabla_i f(x^k)|}{\sqrt{L_i}},$$

we obtain the tightest possible bound on (21). We define the following norm,

$$||x||_L = \sum_{i=1}^n \sqrt{L_i} |x_i|, \tag{22}$$

which has a dual norm of

$$||x||_L^* = \max_i \frac{1}{\sqrt{L_i}} |x_i|.$$

Under this notation, and using the GSL rule, (21) becomes

$$f(x^{k+1}) \le f(x^k) - \frac{1}{2} (\|\nabla f(x^k)\|_L^*)^2,$$

Measuring strong-convexity in the norm $\|\cdot\|_L$ we get

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu_L}{2} ||y - x||_L^2.$$

Minimizing both sides with respect to y we get

$$f(x^*) \ge f(x) - \sup_{y} \{ \langle -\nabla f(x), y - x \rangle - \frac{\mu_L}{2} \|y - x\|_L^2 \}$$

$$= f(x) - \left(\frac{\mu_L}{2} \|\cdot\|_L^2\right)^* (-\nabla f(x))$$

$$= f(x) - \frac{1}{2\mu_L} (\|\nabla f(x)\|_L^*)^2.$$

Putting these together yields

$$f(x^{k+1}) - f(x^*) \le (1 - \mu_L)[f(x^k) - f(x^*)]. \tag{23}$$

Appendix F Comparing μ_L to μ_1 and μ

By the logic Appendix B, to establish a relationship between different strong-convexity constants under different norms, it is sufficient to establish the relationships between the squared norms. In this section, we use this to establish the relationship between μ_L defined in (22) and both μ_1 and μ .

F.1 Relationship between μ_L and μ_1

We have

$$c||x||_1 - ||x||_L = c\sum_i |x_i| - \sum_i \sqrt{L_i}|x_i| = \sum_i (c - \sqrt{L_i})|x_i|,$$

Assuming $c \geq \sqrt{L}$, where $L = \max_i \{L_i\}$, the expression is non-negative and we get

$$||x||_L \le \sqrt{L} ||x||_1.$$

By using

$$c||x||_L - ||x||_1 = \sum_i (c\sqrt{L_i} - 1)|x_i|,$$

and assuming $c \ge \frac{1}{\sqrt{L_{min}}}$, where $L_{min} = \min_i \{L_i\}$, this expression is nonnegative and we get

$$||x||_1 \le \frac{1}{\sqrt{L_{min}}} ||x||_L.$$

The relationship between μ_L and μ_1 is based on the squared norm, so in summary we have

$$\frac{\mu_1}{L} \le \mu_L \le \frac{\mu_1}{L_{min}}.$$

F.2 Relationship between μ_L and μ

Let \vec{L} denote a vector with elements $\sqrt{L_i}$, and we note that

$$\|\vec{L}\| = \left(\sum_i (\sqrt{L_i})^2\right)^{1/2} = \left(\sum_i L_i\right)^{1/2} = \sqrt{n\bar{L}}, \text{ where } \bar{L} = \frac{1}{n}\sum_i L_i.$$

Using this, we have

$$||x||_L = x^T(\operatorname{sign}(x) \circ \vec{L}) \le ||x|| ||\operatorname{sign}(x) \circ \vec{L}|| = \sqrt{n\bar{L}} ||x||.$$

This implies that

$$\frac{\mu}{n\bar{L}} \le \mu_L.$$

Note that we can also show that $\mu_L \leq \frac{\mu}{L_{min}}$, but this is less tight than the upper bound from the previous section because $\mu_1 \leq \mu$.

Appendix G Approximate Gauss-Southwell with additive error

In the additive error regime, the approximate Gauss-Southwell rule chooses an i_k satisfying

$$|\nabla_{i_k} f(x^k)| \ge ||\nabla f(x^k)||_{\infty} - \epsilon_k$$
, where $\epsilon_k \ge 0 \ \forall k$,

and we note that we can assume $\epsilon_k \leq \|\nabla f(x^k)\|_{\infty}$ without loss of generality because we must always choose an i with $|\nabla_{i_k} f(x^k)| \geq 0$. Applying this to our bound on the iteration progress, we get

$$f(x^{k+1}) \leq f(x^{k}) - \frac{1}{2L} \left[\nabla_{i_{k}} f(x^{k}) \right]^{2}$$

$$\leq f(x^{k}) - \frac{1}{2L} \left(\|\nabla f(x^{k})\|_{\infty} - \epsilon_{k} \right)^{2}$$

$$= f(x^{k}) - \frac{1}{2L} \left(\|\nabla f(x^{k})\|_{\infty}^{2} - 2\epsilon_{k} \|\nabla f(x^{k})\|_{\infty} + \epsilon_{k}^{2} \right)$$

$$= f(x^{k}) - \frac{1}{2L} \|\nabla f(x^{k})\|_{\infty}^{2} + \frac{\epsilon_{k}}{L} \|\nabla f(x^{k})\|_{\infty} - \frac{\epsilon_{k}^{2}}{2L}$$
(24)

We first give a result that assumes f is L_1 -Lipschitz continuous in the 1-norm. This implies an inequality that we prove next, followed by a convergence rate that depends on L_1 . However, note that $L \leq L_1 \leq Ln$, so this potentially introduces a dependency on n. We subsequently give a slightly less concise result that has a worse dependency on ϵ but does not rely on L_1 .

G.1 Gradient bound in terms of L_1

We say that ∇f is L_1 -Lipschitz continuous in the 1-norm if we have for all x and y that

$$\|\nabla f(x) - \nabla f(y)\|_{\infty} \le L_1 \|x - y\|_1.$$

Similar to Nesterov [2004, Theorem 2.1.5], we now show that this implies

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L_1} \| \nabla f(y) - \nabla f(x) \|_{\infty}^2,$$
 (25)

and subsequently that

$$\|\nabla f(x^k)\|_{\infty} = \|\nabla f(x^k) - \nabla f(x^*)\|_{\infty} \le \sqrt{2L_1(f(x^k) - f(x^*))} \le \sqrt{2L_1(f(x^0) - f(x^*))},\tag{26}$$

where we have used that $f(x^k) \leq f(x^{k-1})$ for all k and any choice of i_{k-1} (this follows from the basic bound on the progress of coordinate descent methods).

We first show that ∇f being L_1 -Lipschitz continuous in the 1-norm implies that

$$f(y) \le f(x) + \langle \nabla f(x), y - x \rangle + \frac{L_1}{2} ||y - x||_1^2,$$

for all x and y. Consider the function $g(\tau) = f(x + \tau(y - x))$ with $\tau \in \mathbb{R}$. Then

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle = g(1) - g(0) - \langle \nabla f(x), y - x \rangle$$

$$= \int_0^1 \frac{dg}{d\tau}(\tau) - \langle \nabla f(x), y - x \rangle d\tau$$

$$= \int_0^1 \langle \nabla f(x + \tau(y - x)), y - x \rangle - \langle \nabla f(x), y - x \rangle d\tau$$

$$= \int_0^1 \langle \nabla f(x + \tau(y - x)) - \nabla f(x), y - x \rangle d\tau$$

$$\leq \int_0^1 \|\nabla f(x + \tau(y - x)) - \nabla f(x)\|_{\infty} \|y - x\|_1 d\tau$$

$$\leq \int_0^1 L_1 \tau \|y - x\|_1^2 d\tau$$

$$= \frac{L_1}{2} \tau^2 \|y - x\|_1^2 \Big|_0^1$$

$$= \frac{L_1}{2} \|y - x\|_1^2.$$

To subsequently show (25), fix $x \in \mathbb{R}^n$ and consider the function

$$\phi(y) = f(y) - \langle \nabla f(x), y \rangle,$$

which is convex on \mathbb{R}^n and also has an L_1 -Lipschitz continuous gradient in the 1-norm, as

$$\|\phi'(y) - \phi'(x)\|_{\infty} = \|(\nabla f(y) - \nabla f(x)) - (\nabla f(x) - \nabla f(x))\|_{\infty}$$
$$= \|\nabla f(y) - \nabla f(x)\|_{\infty}$$
$$< L_1 \|y - x\|_1.$$

As the minimizer of ϕ is x (i.e., $\phi'(x) = 0$), for any $y \in \mathbb{R}^n$ we have

$$\phi(x) = \min_{v} \phi(v) \le \min_{v} \phi(y) + \langle \phi'(y), v - y \rangle + \frac{L_{1}}{2} \|v - y\|_{1}^{2}$$

$$= \phi(y) - \sup_{v} \langle -\phi'(y), v - y \rangle - \frac{L_{1}}{2} \|v - y\|_{1}^{2}$$

$$= \phi(y) - \frac{1}{2L_{1}} \|\phi'(y)\|_{\infty}^{2}.$$

Substituting in the definition of ϕ , we have

$$f(x) - \langle \nabla f(x), x \rangle \le f(y) - \langle \nabla f(x), y \rangle - \frac{1}{2L_1} \| \nabla f(y) - \nabla f(x) \|_{\infty}^{2}$$

$$\iff f(x) \le f(y) + \langle \nabla f(x), x - y \rangle - \frac{1}{2L_1} \| \nabla f(y) - \nabla f(x) \|_{\infty}^{2}$$

$$\iff f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L_1} \| \nabla f(y) - \nabla f(x) \|_{\infty}^{2}.$$

G.2 Additive error bound in terms of L_1

Using (26) in (24) and noting that $\epsilon_k \geq 0$, we obtain

$$\begin{split} f(x^{k+1}) &\leq f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \frac{\epsilon_k}{L} \|\nabla f(x^k)\|_{\infty} - \frac{\epsilon_k^2}{2L} \\ &\leq f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \frac{\epsilon_k}{L} \sqrt{2L_1(f(x^0) - f(x^*))} - \frac{\epsilon_k^2}{2L} \\ &\leq f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \epsilon_k \frac{\sqrt{2L_1}}{L} \sqrt{f(x^0) - f(x^*)}. \end{split}$$

Applying strong convexity (taken with respect to the 1-norm), we get

$$f(x^{k+1}) - f(x^*) \le \left(1 - \frac{\mu_1}{L}\right) \left[f(x^k) - f(x^*)\right] + \epsilon_k \frac{\sqrt{2L_1}}{L} \sqrt{f(x^0) - f(x^*)},$$

which implies

$$f(x^{k+1}) - f(x^*) \le \left(1 - \frac{\mu_1}{L}\right)^k \left[f(x^0) - f(x^*)\right] + \sum_{i=1}^k \left(1 - \frac{\mu_1}{L}\right)^{k-i} \epsilon_i \frac{\sqrt{2L_1}}{L} \sqrt{f(x^0) - f(x^*)}$$
$$= \left(1 - \frac{\mu_1}{L}\right)^k \left[f(x^0) - f(x^*) + \sqrt{f(x^0) - f(x^*)}A_k\right],$$

where

$$A_k = \frac{\sqrt{2L_1}}{L} \sum_{i=1}^k \left(1 - \frac{\mu_1}{L}\right)^{-i} \epsilon_i.$$

G.3 Additive error bound in terms of L

By our additive error inequality, we have

$$|\nabla_{i_k} f(x^k)| + \epsilon_k \ge ||\nabla f(x^k)||_{\infty}.$$

Using this again in (24) we get

$$f(x^{k+1}) \leq f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \frac{\epsilon_k}{L} \|\nabla f(x^k)\|_{\infty} - \frac{\epsilon_k^2}{2L}$$

$$\leq f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \frac{\epsilon_k}{L} (|\nabla_{i_k} f(x^k)| + \epsilon_k) - \frac{\epsilon_k^2}{2L}$$

$$= f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \frac{\epsilon_k}{L} |\nabla_{i_k} f(x^k)| + \frac{\epsilon_k^2}{2L}.$$

Further, from our basic progress bound that holds for any i_k we have

$$f(x^*) \leq f(x^{k+1}) \leq f(x^k) - \frac{1}{2L} \left[\nabla_{i_k} f(x^k) \right]^2 \leq f(x^0) - \frac{1}{2L} \left[\nabla_{i_k} f(x^k) \right]^2,$$

which implies

$$|\nabla_{i_k} f(x^k)| \le \sqrt{2L(f(x^0) - f(x^*))}.$$

and thus that

$$f(x^{k+1}) \le f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \frac{\epsilon_k}{L} \sqrt{2L(f(x^0) - f(x^*))} + \frac{\epsilon_k^2}{2L}$$
$$= f(x^k) - \frac{1}{2L} \|\nabla f(x^k)\|_{\infty}^2 + \epsilon_k \sqrt{\frac{2}{L}} \sqrt{f(x^0) - f(x^*)} + \frac{\epsilon_k^2}{2L}.$$

Applying strong convexity and applying the inequality recursively we obtain

$$f(x^{k+1}) - f(x^*) \le \left(1 - \frac{\mu_1}{L}\right)^k \left[f(x^0) - f(x^*)\right] + \sum_{i=1}^k \left(1 - \frac{\mu_1}{L}\right)^{k-i} \left(\epsilon_i \sqrt{\frac{2}{L}} \sqrt{f(x^0) - f(x^*)} + \frac{\epsilon_i^2}{2L}\right)$$

$$= \left(1 - \frac{\mu_1}{L}\right)^k \left[f(x^0) - f(x^*) + A_k\right],$$

where

$$A_k = \sum_{i=1}^k \left(1 - \frac{\mu_1}{L}\right)^{-i} \left(\sqrt{\frac{2}{L}} \epsilon_i \sqrt{f(x^0) - f(x^*)} + \frac{\epsilon_i^2}{2L}\right).$$

Although uglier than the expression depending on L_1 , this expression will tend to be smaller unless ϵ_k is not small.

Appendix H Convergence Analysis of GS-s, GS-r, and GS-q Rules

In this section, we consider problems of the form

$$\min_{x \in \mathbb{R}^n} F(x) = f(x) + g(x) = f(x) + \sum_{i=1}^n g_i(x_i),$$

where f satisfies our usual assumptions, but the g_i can be non-smooth. We first introduce some notation that will be needed to state our result for the GS-q rule, followed by stating the result and then showing that it holds in two parts. We then turn to showing that the rule cannot hold in general for the GS-s and GS-r rules.

H.1 Notation and basic inequality

To analyze this case, an important inequality we will use is that the L-Lipschitz-continuity of $\nabla_i f$ implies that for all x, i, and d that

$$F(x + de_i) = f(x + de_i) + g(x + de_i) \le f(x) + \langle \nabla f(x), de_i \rangle + \frac{L}{2} d^2 + g(x + de_i)$$

$$= f(x) + g(x) + \langle \nabla f(x), de_i \rangle + \frac{L}{2} d^2 + g_i(x_i + d) - g_i(x_i)$$

$$= F(x) + V_i(x, d),$$
(27)

where

$$V_i(x,d) \equiv \langle \nabla f(x), de_i \rangle + \frac{L}{2}d^2 + g_i(x_i + d) - g_i(x_i).$$

Notice that the GS-q rule is defined by

$$i_k = \underset{i}{\operatorname{argmin}} \{ \underset{d}{\operatorname{min}} V_i(x, d) \}.$$

We use the notation $d_i^k = \operatorname{argmin}_d V_i(x^k, d)$ and we will use d^k to denote the vector containing these values for all i. When using the GS-q rule, the iteration is defined by

$$x^{k+1} = x^k + d_{i_k} e_{i_k}$$

= $x^k + \underset{d}{\operatorname{argmin}} \{ V_{i_k}(x, d) \} e_{i_k}.$ (28)

In this notation the GS-r rule is given by

$$j_k = \operatorname*{argmax}_{i} |d_i^k|.$$

We will use the notation x_+^k to be the step that would be taken at x_k if we update coordinate j_k according the GS-r rule

$$x_+^k = x^k + d_{j_k} e_{j_k}.$$

From the optimality of d_i^k , we have for any i that

$$-L[(x_i^k - \frac{1}{L}\nabla_i f(x^k)) - (x_i^k + d_i^k)] \in \partial g_i(x_i^k + d_i^k), \tag{29}$$

and we will use the notation s_j^k for the unique element of $\partial g_j(x_j^k + d_j^k)$ satisfying this relationship. We use s^k to denote the vector containing these values.

H.2 Convergence bound for GS-q rule

Under this notation, we can show that coordinate descent with the GS-q rule satisfies the bound

$$F(x^{k+1}) - F(x^*) \le \min\left\{ \left(1 - \frac{\mu}{Ln}\right) [f(x^k) - f(x^*)], \left(1 - \frac{\mu_1}{L}\right) [f(x^k) - f(x^*)] + \epsilon_k \right\},\tag{30}$$

where

$$\epsilon_k \le \frac{\mu_1}{L} \left(g(x_+^k) - g(x^k + d^k) + \langle s^k, (x^k + d^k) - x_+^k \rangle \right).$$

We note that if g is linear then $\epsilon^k = 0$ and this convergence rate reduces to

$$F(x^{k+1}) - F(x^*) \le \left(1 - \frac{\mu_1}{L}\right) \left[F(x^k) - F(x^*)\right].$$

Otherwise, ϵ_k depends on how far $g(x_+^k)$ lies above a particular linear underestimate extending from $(x^k + d^k)$, as well as the conditioning of f. We show this result by first showing that the GS-q rule makes at least as much progress as randomized selection (first part of the min), and then showing that the GS-q rule also makes at least as much progress as the GS-r rule (second part of the min).

H.3 GS-q is at least as fast as random

Our argument in this section follows a similar approach to Richtárik and Takáč [2014]. In particular, combining (27) and (28) we have the following upper bound on the iteration progress

$$\begin{split} F(x^{k+1}) &\leq F(x^k) + \min_{i \in \{1, 2, \dots, n\}} \left\{ \min_{d \in \mathbb{R}} V_i(x^k, d) \right\}, \\ &= F(x^k) + \min_{i \in \{1, 2, \dots, n\}} \left\{ \min_{y \in \mathbb{R}^n} V_i(x^k, y_i - x_i^k) \right\}, \\ &= F(x^k) + \min_{y \in \mathbb{R}^n} \left\{ \min_{i \in \{1, 2, \dots, n\}} V_i(x^k, y_i - x_i^k) \right\}, \\ &\leq F(x^k) + \min_{y \in \mathbb{R}^n} \left\{ \frac{1}{n} \sum_{i=1}^n V_i(x^k, y_i - x^k) \right\} \\ &= F(x^k) + \frac{1}{n} \min_{y \in \mathbb{R}^n} \left\{ \langle \nabla f(x^k), y - x^k \rangle + \frac{L}{2} \|y - x^k\|^2 + g(y) - g(x^k) \right\} \\ &= \left(1 - \frac{1}{n} \right) F(x^k) + \frac{1}{n} \min_{y \in \mathbb{R}^n} \left\{ f(x^k) + \langle \nabla f(x^k), y - x^k \rangle + \frac{L}{2} \|y - x^k\|^2 + g(y) \right\}. \end{split}$$

From strong convexity of f, we have that F is also μ -strongly convex and that

$$f(x^k) \le f(y) - \langle \nabla f(x^k), y - x^k \rangle - \frac{\mu}{2} \|y - x^k\|^2,$$
$$F(\alpha x^* + (1 - \alpha)x^k) \le \alpha F(x^*) + (1 - \alpha)F(x^k) - \frac{\alpha(1 - \alpha)\mu}{2} \|x^k - x^*\|^2,$$

for any $y \in \mathbb{R}^n$ and any $\alpha \in [0,1]$ [see Nesterov, 2004, Theorem 2.1.9]. Using these gives us

$$\begin{split} &F(x^{k+1})\\ &\leq \left(1-\frac{1}{n}\right)\!F(x^k) + \frac{1}{n}\min_{y\in\mathbb{R}^n}\left\{f(y) - \frac{\mu}{2}\|y-x\|^2 + \frac{L}{2}\|y-x^k\|^2 + g(y)\right\}\\ &= \left(1-\frac{1}{n}\right)\!F(x^k) + \frac{1}{n}\min_{y\in\mathbb{R}^n}\left\{F(y) + \frac{L-\mu}{2}\|y-x^k\|^2\right\}\\ &\leq \left(1-\frac{1}{n}\right)\!F(x^k) + \frac{1}{n}\min_{\alpha\in[0,1]}\left\{F(\alpha x^* + (1-\alpha)x^k) + \frac{\alpha^2(L-\mu)}{2}\|x^k-x^*\|^2\right\}\\ &\leq \left(1-\frac{1}{n}\right)\!F(x^k) + \frac{1}{n}\min_{\alpha\in[0,1]}\left\{\alpha F(x^*) + (1-\alpha)F(x^k) + \frac{\alpha^2(L-\mu)-\alpha(1-\alpha)\mu}{2}\|x^k-x^*\|^2\right\}\\ &\leq \left(1-\frac{1}{n}\right)\!F(x^k) + \frac{1}{n}\!\left[\alpha^*F(x^*) + (1-\alpha^*)F(x^k)\right] \qquad \left(\text{choosing }\alpha^* = \frac{\mu}{L}\in(0,1]\right)\\ &= \left(1-\frac{1}{n}\right)\!F(x^k) + \frac{\alpha^*}{n}F(x^*) + \frac{(1-\alpha^*)}{n}F(x^k)\\ &= F(x^k) - \frac{\alpha^*}{n}[F(x^k) - F(x^*)]. \end{split}$$

Subtracting $F(x^*)$ from both sides of this inequality gives us

$$F(x^{k+1}) - F(x^*) \le \left(1 - \frac{\mu}{nL}\right) [F(x^k) - F(x^*)].$$

H.4 GS-q is at least as fast as GS-r

In this section we derive the right side of the bound (30) for the GS-r rule, but note it also applies to the GS-q rule because from (27) and (28) we have

$$F(x^{k+1}) \leq F(x^k) + \min_{i} V_i(x, d_i^k)$$
 (GS-q rule)
 $\leq F(x^k) + V_{j_k}(x, d_{j_k}^k)$ (j_k selected by the GS-r rule).

Note that we lose progress by considering a bound based on the GS-r rule, but its connection to the ∞ -norm will make it easier to derive an upper bound.

By the convexity of g_{j_k} we have

$$g_{j_k}(x_{j_k}^k) \ge g_{j_k}(x_{j_k}^k + d_{j_k}^k) + s_{j_k}^k(x_{j_k}^k - (x_{j_k}^k + d_{j_k}^k))$$

$$= g_{j_k}(x_{j_k}^k + d_{j_k}^k) - (-Ld_{j_k}^k - \nabla_{j_k}f(x^k))(d_{j_k}^k)$$

$$= g_{j_k}(x_{j_k}^k + d_{j_k}^k) + \nabla_{j_k}f(x^k)d_{j_k}^k + L(d_{j_k}^k)^2,$$

where s_i^k is defined by (29). Using this we have that

$$\begin{split} F(x^{k+1}) &\leq F(x^k) + V_j(x, d_{j_k}^k) \\ &= F(x^k) + \nabla_j f(x^k) (d_{j_k}^k) + \frac{L}{2} (d_{j_k}^k)^2 + g_i (x_{j_k}^k + d_{j_k}^k) - g_i (x_{j_k}^k) \\ &\leq F(x^k) + \nabla_j f(x^k) (d_{j_k}^k) + \frac{L}{2} (d_{j_k}^k)^2 - \nabla_{j_k} f(x^k) d_{j_k}^k - L(d_{j_k}^k)^2 \\ &= F(x^k) - \frac{L}{2} (d_{j_k}^k)^2. \end{split}$$

Adding and subtracting $F(x^*)$ and noting that j_k is selected using the GS-r rule, we obtain the upper bound

$$F(x^{k+1}) - F(x^*) \le F(x^k) - F(x^*) - \frac{L}{2} ||d^k||_{\infty}^2.$$
(31)

Recall that we use x_+^k to denote the iteration that would result if we chose j_k and actually performed the GS-r update. Using the Lipschitz continuity of the gradient and definition of the GS-q rule again, we have

$$F(x^{k+1}) \leq F(x^k) + \nabla f(x^k)^T (x^{k+1} - x^k) + \frac{L}{2} ||x^{k+1} - x^k||^2 + g(x^{k+1}) - g(x^k)$$

$$\leq F(x^k) + \nabla f(x^k)^T (x_+^k - x^k) + \frac{L}{2} ||x_+^k - x^k||^2 + g(x_k^+) - g(x^k)$$

$$= f(x^k) + \nabla f(x^k)^T (x_+^k - x^k) + \frac{L}{2} ||d^k||_{\infty}^2 + g(x_+^k)$$

By the strong-convexity of f, for any $y \in \mathbb{R}^N$ we have

$$f(x^k) \le f(y) - \nabla f(x^k)^T (y - x^k) - \frac{\mu_1}{2} ||y - x^k||_1^2,$$

and using this we obtain

$$F(x^{k+1}) \le f(y) + \nabla f(x^k)^T (x_+^k - y) - \frac{\mu_1}{2} \|y - x^k\|_1^2 + \frac{L}{2} \|d^k\|_{\infty}^2 + g(x_+^k). \tag{32}$$

By the convexity of g and $s^k \in \partial g(x^k + d^k)$, we have

$$g(y) \ge g(x^k + d^k) + \langle s^k, y - (x^k + d^k) \rangle.$$

Combining (32) with the above inequality, we have

$$F(x^{k+1}) - F(y) \le \langle \nabla f(x^k), x_+^k - y \rangle - \frac{\mu_1}{2} \|y - x^k\|_1^2 + \frac{L}{2} \|d^k\|_{\infty}^2 + g(x_+^k) - g(x^k + d^k) + \langle s^k, (x^k + d^k) - y \rangle.$$

We add and subtract $\langle s^k, x_+^k \rangle$ on the right-hand side to get

$$F(x^{k+1}) - F(y) \le \langle \nabla f(x^k) + s^k, x_+^k - y \rangle - \frac{\mu_1}{2} \|y - x^k\|_1^2 + \frac{L}{2} \|d^k\|_{\infty}^2 + g(x_+^k) - g(x_+^k) + \langle s^k, (x^k + d^k) - x_+^k \rangle.$$

Let $c^k = g(x_+^k) - g(x^k + d^k) + \langle s^k, (x^k + d^k) - x_+^k \rangle$, which is non-negative by the convexity g. Making this substitution, we have

$$F(y) \ge F(x^{k+1}) + \langle -Ld^k, y - x_+^k \rangle + \frac{\mu_1}{2} \|y - x^k\|_1^2 - \frac{L}{2} \|d^k\|_{\infty}^2 - c^k.$$

Now add and subtract $\langle -Ld^k, x^k \rangle$ to the right-hand side and use (29) to get

$$F(y) \ge F(x^{k+1}) + \langle -Ld^k, y - x^k \rangle + \frac{\mu_1}{2} \|y - x^k\|_1^2 - \frac{L}{2} \|d^k\|_{\infty}^2 - L\langle d^k, x^k - x_+^k \rangle - c^k.$$

Minimizing both sides with respect to y results in

$$F(x^*) \ge F(x^{k+1}) - \frac{L^2}{2\mu_1} \|d^k\|_{\infty}^2 - \frac{L}{2} \|d^k\|_{\infty}^2 - L\langle d^k, x^k - x_+^k \rangle - c^k$$

$$\ge F(x^{k+1}) - \frac{L^2}{2\mu_1} \|d^k\|_{\infty}^2 - \frac{L}{2} \|d^k\|_{\infty}^2 + L\|d^k\|_{\infty}^2 - c^k$$

$$= F(x^{k+1}) - \frac{L(L - \mu_1)}{2\mu_1} \|d^k\|_{\infty}^2 - c^k,$$

where we have used that $x_+^k = x^k + d_{j_k}^k e_{j_k}$ and $|d_{j_k}^k| = ||d^k||_{\infty}$. Combining this with equation (31), we get

$$F(x^{k+1}) - F(x^*) \le F(x^k) - F(x^*) - \frac{L}{2} \|d^k\|_{\infty}^2$$

$$F(x^{k+1}) - F(x^*) \le F(x^k) - F(x^*) - \frac{\mu_1}{(L - \mu_1)} \left[F(x^{k+1}) - F(x^*) - c^k \right]$$

$$\left(1 + \frac{\mu_1}{(L - \mu_1)} \right) \left[F(x^{k+1}) - F(x^*) \right] \le F(x^k) - F(x^*) + \epsilon^k \frac{\mu_1}{(L - \mu_1)}$$

$$F(x^{k+1}) - F(x^*) \le \frac{(L - \mu_1)}{L} \left[F(x^k) - F(x^*) \right] + c^k \frac{\mu_1}{L}$$

$$F(x^{k+1}) - F(x^*) \le \left(1 - \frac{\mu_1}{L} \right) \left[F(x^k) - F(x^*) \right] + c^k \frac{\mu_1}{L}.$$

H.5 Lack of progress of the GS-s rule

We now show that the rate $(1 - \mu_1/L)$, and even the slower rate $(1 - \mu/Ln)$, cannot hold for the GS-s rule. We do this by constructing a problem where an iteration of the GS-s method does not make sufficient progress. In particular, consider the bound-constrained problem

$$\min_{x \in C} f(x) = \frac{1}{2} ||Ax - b||_2^2,$$

where $C = \{x : x \ge 0\}$, and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \quad x^0 = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}, \quad x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We thus have that

$$f(x^{0}) = \frac{1}{2}((1+1)^{2} + (.07+3)^{2}) \approx 6.7$$

$$f(x^{*}) = \frac{1}{2}((-1)^{2} + (-3)^{2}) = 5$$

$$\nabla f(x^{0}) = A^{T}(Ax_{0} - b) \approx \begin{pmatrix} 2.0\\2.1 \end{pmatrix}$$

$$\nabla^{2} f(x) = A^{T} A = \begin{pmatrix} 1 & 0\\0 & 0.49 \end{pmatrix}.$$

The parameter values for this problem are

$$n = 2$$
 $\mu = \lambda_{min} = 0.49$
 $L = \lambda_{max} = 1$
 $\mu_1 = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^{-1} = 1 + \frac{1}{0.49} \approx 0.33,$

where the λ_i are the eigenvalues of $A^T A$, and μ and μ_1 are the corresponding strong-convexity constants for the 2-norm and 1-norm, respectively.

The proximal operator of the indicator function is the projection onto the set C, which involves setting negative elements to zero. Thus, our iteration update is given by

$$x^{k+1} = \max_{\delta_C} [x^k - \frac{1}{L} \nabla_{i_k} f(x^k) e_{i_k}] = \max(x^k - \frac{1}{L} \nabla_{i_k} f(x^k) e_{i_k}, 0),$$

For this problem, the GS-s rule is given by

$$i = \underset{i}{\operatorname{argmax}} |\eta_i^k|,$$

where

$$\eta_i^k = \begin{cases} \nabla_i f(x^k), & \text{if } x_i^k \neq 0 \text{ or } \nabla_i f(x^k) < 0 \\ 0, & \text{otherwise} \end{cases}.$$

Based on the value of $\nabla f(x^0)$, the GS-s rule thus chooses to update coordinate 2, setting it to zero and obtaining

$$f(x^1) = \frac{1}{2}((1+1)^2 + (-3)^2) = 6.5.$$

Thus we have

$$\frac{f(x^1) - f(x^*)}{f(x^0) - f(x^*)} \approx \frac{6.5 - 5}{6.7 - 5} \approx 0.88,$$

even though the bounds obtain the faster rates of

$$\left(1 - \frac{\mu}{Ln}\right) = \left(1 - \frac{0.49}{2}\right) \approx 0.76,$$

 $\left(1 - \frac{\mu_1}{L}\right) \approx (1 - 0.33) = 0.67.$

Thus, the GS-s rule does not satisfy either bound. On the other hand, the GS-r and GS-q rules are given in this context by

$$i_k = \underset{i}{\operatorname{argmax}} \left| \max \left(x^k - \frac{1}{L} \nabla_i f(x^k) e_i, 0 \right) - x^k \right|,$$

and thus both these rules choose to update coordinate 1, setting it to zero to obtain $f(x^1) \approx 5.2$ and a progress ratio of

$$\frac{f(x^1) - f(x^*)}{f(x^0) - f(x^*)} \approx \frac{5.2 - 5}{6.7 - 5} \approx 0.12,$$

which clearly satisfies both bounds.

H.6 Lack of progress of the GS-r rule

We now turn to showing that the GS-r rule does not satisfy these bounds in general. It will not be possible to show this for a simple bound-constrained problem since the GS-r and GS-q rules are equivalent for these problems. Thus, we consider the following ℓ_1 -regularized problem

$$\min_{x \in \mathbb{R}^2} \frac{1}{2} ||Ax - b||_2^2 + \lambda ||x||_1 \equiv F(x).$$

We use the same A as the previous section, so that n, μ, L , and μ_1 are the same. However, we now take

$$b = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 0.4 \\ 0.5 \end{pmatrix}, \quad x_* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \lambda = 1,$$

so we have

$$f(x_0) \approx 3.1, \quad f(x_*) = 2$$

The proximal operator of the absolute value function is given by the soft-threshold function, and our coordinate update of variable i_k is given by

$$x_{i_k}^{k+1} = \max_{\lambda \mid \cdot \mid} [x_{i_k}^{k+\frac{1}{2}}] = \operatorname{sgn}(x_{i_k}^{k+\frac{1}{2}}) \cdot \max(x_{i_k}^{k+\frac{1}{2}} - \lambda/L, 0),$$

where we have used the notation

$$x_i^{k+\frac{1}{2}} = x_i^k - \frac{1}{L} \nabla_i f(x^k) e_i.$$

The GS-r rule is defined by

$$i_k = \operatorname*{argmax}_{i} |d_i^k|,$$

where $d_i^k = \operatorname{prox}_{\lambda|\cdot|}[x_i^{k+\frac{1}{2}}] - x_i^k$ and in this case

$$d^0 = \begin{pmatrix} 0.6 \\ -0.5 \end{pmatrix}.$$

Thus, the GS-r rule chooses to update coordinate 1. After this update the function value is

$$F(x^1) \approx 2.9$$
,

so the progress ratio is

$$\frac{F(x^1) - F(x^*)}{F(x^0) - F(x^*)} \approx \frac{2.9 - 2}{3.1 - 2} \approx 0.84.$$

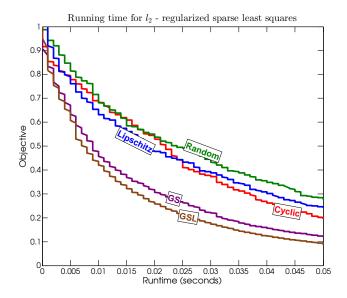


Figure 3: Comparison of coordinate selection rules for ℓ_2 -regularized sparse least squares.

However, the bounds suggest faster progress ratios of

$$\left(1 - \frac{\mu}{Ln}\right) \approx 0.76,$$

$$\left(1 - \frac{\mu_1}{L}\right) \approx 0.67,$$

so the GS-r rule does not satisfy either bound. In contrast, in this setting the GS-q rule chooses to update coordinate 2 and obtains $F(x^1) \approx 2.2$, obtaining a progress ratio of

$$\frac{F(x^1) - F(x^*)}{F(x^0) - F(x^*)} \approx \frac{2.2 - 2}{3.1 - 2} \approx 0.16,$$

which satisfies both bounds by a substantial margin. Indeed, we used a genetic algorithm to search for a setting of the parameters of this problem (values of x^0 , λ , b, and the diagonals of A) that would make the GS-q not satisfy the bound depending on μ_1 , and it easily found counter-examples for the GS-s and GS-r rules but was not able to produce a counter example for the GS-q rule.

Appendix I Runtime Experiments

In Figure 3 we plot the objective against the runtime for the ℓ_2 -regularized sparse least squares problem from our experiments. Although runtimes are very sensitive to exact implementation details and we believe that more clever implementations than our naive Python script are possible, this figure does show that the GS and GSL rules offer benefits in terms of runtime with our implementation and test hardware.

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